SEM-4 (MATHEMATICS GENERAL)

ANUP RAY

MIKADO Page No. Date

Theorem 1: If H be a sub-group of a group G and $h \in H$, then $hH = Hh = H$.

Proof: Let h' be an arbitrary element of H, so that from definition of left coset, we have

$hh' \in hH$.

Since H is a subgroup, we have $h \in H$, $h' \in H \Rightarrow hh' \in H$.

Therefore every element of hH is also an element of H and we have $hH \subset H$ --- (1)

Again $h' = (hh^{-1})h' = h(h^{-1}h') \in hH$, since $h^{-1} \in H$, $h' \in H \Rightarrow h^{-1}h' \in H$.

We conclude then every element of H is also an element of hH. Hence $H \subset hH$ -- (2)

From (1) & (2), it follows that $hH = H$.

Similarly, it can be shown that $Hh = H$

Theorem 2: Any two left cosets of a sub-group are either disjoint or identical.

Proof: Let H be a subgroup of a group G and let aH and bH be two of its left cosets.

Let us assume $aH \cap bH \neq \phi$ and let c be the common element of the two cosets.

Then we can write $c = ah$ and $c = bh'$, for $h, h' \in H$.

Therefore $ah = bh'$, giving $a = bh'h^{-1}$

Since H is a subgroup, we have $h'h^{-1} \in H$.

Let $h'h^{-1} = h''$ so that $a = bh''$

Hence $aH = (bh'')H = b(h''H) = bH$, since $h''H = H$.

Hence two left cosets aH and bH are identical, if $aH \cap bH = \phi$

Thus either $aH \cap bH = \phi$ or $aH = bH$

Theorem 3: Any two right cosets of a sub-group are either disjoint or identical.

Proof: Left for students. (Same as theorem 2)

Theorem 4: If H be a sub-group of a group G and $a, b \in G$, then

(i) $aH = H$ if and only if $a \in H$.

(ii) $Ha = H$ if and only if $a \in H$.

(iii) $aH = bH$ if and only if $a^{-1}b \in H$

(iv) $Ha = Hb$ if and only if $ba^{-1} \in H$.

Proof: (i) Let $aH = H$. Then $a = ae \in aH = H$. Conversely, let $a \in H$. Then for any $h \in H$, $h = eh = aa^{-1}h \in aH \Rightarrow H \subseteq aH$. Since H is a subgroup and $a \in H$, we find that $aH = \{ah \mid h \in H\} \subseteq H$. Hence $H = aH$.

(ii) Proof is similar to (i)

(iii) Let $aH = bH$. Since $b = be \in bH = aH$, there exists $h \in H$ such that $b = ah$ for some $h \in H$. Then $a^{-1}b = h \in H$.

Conversely, let $a^{-1}b \in H$. Hence

$bH = aa^{-1}bH = a(a^{-1}bH) = aH$, since by (i), $a^{-1}b \in H \Rightarrow a^{-1}bH = H$.

(iv) Proof is similar to (iii)